The capacity of sets of divergence of certain Taylor series on the unit circle

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Abstract

A simple and direct proof is given of a generalization of a classical result on the convergence of $\sum_{k=0}^{\infty} a_k e^{ikx}$ outside sets of x of an appropriate capacity zero where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in the unit disc U and $\sum_{k=0}^{\infty} k^{\alpha}|a_k|^2 < \infty$ with $\alpha \in (0, 1]$. We also detail some convergence consequences of our results for weighted Besov spaces, for the classes of analytic functions in U for which $\sum_{k=1}^{\infty} k^{\gamma}|a_k|^p < \infty$, and for trigonometric series of the form $\sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx)$ with $\sum_{k=1}^{\infty} k^{\gamma}(|\alpha_k|^p + |\beta_k|^p) < \infty$, where $\gamma > 0$, $p > 1$.

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1 *The class $\mathcal{H}^p_\beta$ and convergence*

We begin by defining the L^p-capacities that we will use to measure exceptional sets. First, let K be a kernel on $\mathbf{R}$, that is, K is a non-negative, even and unbounded integrable function on $\mathbf{R}$ that is decreasing on $(0, \infty)$. We define the convolution $K * F$ of K with any $F \in L^p(\mathbf{R})$, $p > 1$, by

$$K * F(x) \;=\; \int_{\mathbf{R}} K(t)\, F(x - t)\, dt$$

and define $T(K, p, E), E \subset \mathbf{R}$, to be the set of $F \in L^p(\mathbf{R})$ with $F \geq 0$ on $\mathbf{R}$ such that $K * F(x) \geq 1$ for every $x \in E$. Following Meyers [11], we then define the $C_{K,p}$-capacity of E by

$$C_{K,p}(E) = \inf\{\int_{\mathbf{R}} F(x)^p dx \;:\; F \in T(K, p, E)\}.$$

See [11],[1], and [13] for the basic properties of such capacities. We note here that any set of zero $C_{K,p}$-capacity also has Lebesgue measure zero, and also note one further simple fact which is a significant element in our proof of Theorem 1 below: if $F \in L^p(\mathbf{R}), p > 1$, with $F \geq 0$ on $\mathbf{R}$, then $K * F < \infty$ (*i.e.* $K * F$ exists as a finite integral) outside a set $E \in \mathbf{R}$ with $C_{K,p}(E) = 0$, or, equivalently, $K * F$ is finite $C_{K,p}$–*quasieverywhere* in $\mathbf{R}$. To show this we simply 'pick $F_0 \in T(K, p, E)$ and let $E = \{x : K * F_0(x) = \infty\}$. Then $\delta F_0 \in T(K, p, E)$ for every $\delta > 0$ and, letting $\delta \to 0$, we obtain $C_{K,p}(E) = 0$' ([13, p. 342]).

In Theorem 1 we work with the one-dimensional Bessel kernels G_α, $0 < \alpha \leq 1$, a family of kernels that decay exponentially as $|x| \to \infty$ and satisfy the estimates

$$G_\alpha(x) \ \sim \ |x|^{\alpha-1}\,, \ 0 < \alpha < 1\,, \quad G_1(x) \ \sim \ \log\frac{1}{|x|}\,, \tag{1.1}$$

as $x \to 0$, where $u \sim v$ means that u/v is bounded above and below by positive constants for all sufficiently small non-zero $|x|$. An explicit formula for $G_\alpha(x)$ can be found in [2]. We write $C_{\alpha,p}$ for $C_{K,p}$ when $K = G_\alpha$, $0 < \alpha \leq 1$.

We complete the preparations for Theorem 1 by defining a class $\mathcal{H}^p_\beta$ of analytic functions f in $U = \{z : |z| < 1\}$ (see [10],[13],[16]). We write $f \in \mathcal{H}^p_\beta$, where $0 < \beta < 1$ and $p > 1$, if there is a function $F \in L^p(-\pi, \pi)$ such that

$$f(z) \ = \ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(t)\,dt}{(1 - z\mathrm{e}^{-it})^{1-\beta}}\,, \quad z \in U. \tag{1.2}$$

We assume that F is extended by periodicity to $[-3\pi, 3\pi]$ and that $F \equiv 0$ in $\mathbf{R}\backslash[-3\pi, 3\pi]$, so that $F \in L^p(\mathbf{R})$ also. It is easy to show that $\mathcal{H}^2_\beta = D_{2\beta}$ for $\beta \in (0, 1/2]$, where D_α, $0 < \alpha \leq 1$, denotes the Dirichlet-type space of analytic functions $f(z) = \sum_{k=0}^\infty a_k z^k$ in U for which $\sum_{k=0}^\infty k^\alpha |a_k|^2 < \infty$. For if $f \in \mathcal{H}^2_\beta$, and we write $(1 - w)^{\beta-1} = \sum_{k=0}^\infty b_k(\beta) w^k$, $|w| < 1$, then, by (1.2),

$$\sum_{k=0}^\infty a_k z^k = f(z) = \frac{1}{2\pi}\sum_{k=0}^\infty\left(\int_{-\pi}^{\pi}F(t)\mathrm{e}^{-ikt}dt\right)b_k(\beta)z^k = \sum_{k=0}^\infty c_k b_k(\beta) z^k$$

so $a_k = b_k(\beta)c_k$ or $c_k = b_k(\beta)^{-1}a_k$. As $(c_k)_1^\infty$ consists of Fourier coefficients of F, and $F \in L^2(-\pi, \pi)$, we have $\sum_{k=0}^\infty |c_k|^2 < \infty$ and, since [19, vol. 1, p. 77]

$$b_k(\beta) = \frac{k^{-\beta}}{\Gamma(1-\beta)}\{1 + O(\frac{1}{k})\}, \tag{1.3}$$

we thus have $\sum k^{2\beta}|a_k|^2 < \infty$. Conversely, assume $\sum k^{2\beta}|a_k|^2 < \infty$ and that $f(z) = \sum_{k=0}^\infty a_k z^k$. Set $d_k = b_k(\beta)^{-1}a_k$ for $k \geq 0$. Then $\sum_{k=0}^\infty |d_k|^2 < \infty$ and

$(d_k)_{-\infty}^{\infty}$, with $d_k = 0$ for $k < 0$, is the sequence of Fourier coefficients of a function $F \in L^2$ and we have $f \in \mathcal{H}_\beta^2$. The equivalence of $\mathcal{H}_\beta^2$ and $D_{2\beta}$ is shown.

The familiar proofs in [19], [15] of the convergence results for $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ for functions in D_α involve proving that the partial sums of the series are bounded outside exceptional sets of $C_{1-\alpha}$ capacity zero and then deducing the desired convergence result by a simple argument. Our approach is more direct and is based on showing that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges for every x for which $G_\beta * |F|(x)$ is finite, from which the required result follows immediately.

Theorem 1 *Let $f \in \mathcal{H}_\beta^p$ with $0 < \beta < 1, p > 1$, and assume that $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in U$. Then $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges $C_{\beta,p}$-quasieverywhere on $[-\pi,\pi]$ to $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(t)dt}{(1-\mathrm{e}^{-i(t-x)})^{1-\beta}}$.*

Proof of Theorem 1. Let f be related to $F \in L^p(\mathbf{R})$ as in (1.2). Then, by an earlier observation, there is a set $E \subset [-\pi,\pi]$ of $C_{\beta,p}$-capacity zero such that

$$\int_{\mathbf{R}} G_\beta(t)\,|F(x-t)|\,dt < \infty, \quad x \in E' = [-\pi,\pi]\backslash E. \tag{1.4}$$

Let $x_0 \in E'$. To prove Theorem 1 it will be enough to show that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx_0}$ converges to the desired limit. To this end, note that by (1.2), as observed above, $a_k = b_k(\beta)c_k$ where (c_k) is the sequence of Fourier coefficients of F, and so, for $n \geq 1$,

$$\begin{aligned} 2\pi \sum_{k=0}^{n} a_k \mathrm{e}^{ikx_0} &= \int_{-\pi}^{\pi} \left(\sum_{k=0}^{n} b_k(\beta)\mathrm{e}^{ik(x_0-t)}\right) F(t)\,dt \\ &= \int_{-\pi}^{\pi} \left(\sum_{k=0}^{n} b_k(\beta)\mathrm{e}^{ikt}\right) F(x_0-t)dt. \end{aligned} \tag{1.5}$$

Next by (1.3), the inequality $|\sum_{k=1}^{n} k^{-\beta}\mathrm{e}^{ikt}| \leq c_\beta |t|^{\beta-1}, 0 < |t| \leq \pi$ [19, vol.1, p. 191], and (1.1), for each $n \geq 1$,

$$\left|\sum_{k=0}^{n} b_k(\beta)\mathrm{e}^{ikt}\right| \leq c_\beta |t|^{\beta-1} + A_\beta \leq c'_\beta G_\beta(t) + A_\beta, \tag{1.6}$$

where A_β, c_β, and c'_β are positive quantities which depend only on β, so we have, for $0 < |t| \leq \pi$ and $n \geq 1$,

$$\begin{aligned} \left|\left(\sum_{k=0}^{n} b_k(\beta)\mathrm{e}^{ikt)}\right) F(x_0-t)\right| &\leq c'_\beta G_\beta(t)|F(x_0-t)| + A_\beta|F(x_0-t)| \\ &\equiv Q_0(t), \end{aligned} \tag{1.7}$$

where $Q_0 \in L(-\pi,\pi)$ by (1.4). Since, for each fixed $\beta \in (0,1)$, $(b_k(\beta))$ is a decreasing sequence [19, vol. 1, p. 77], it follows (*loc.cit.*, p.4) that $\sum_{k=0}^{\infty} b_k(\beta)\mathrm{e}^{ikt}$ converges for non-zero $t \in [-\pi,\pi]$, and hence, for such t,

$$(1-\mathrm{e}^{it})^{\beta-1} = \lim_{r\to 1}(1-r\mathrm{e}^{it})^{\beta-1} = \lim_{r\to 1}\sum_{k=0}^{\infty} b_k(\beta)\mathrm{e}^{ikt}r^k = \sum_{k=0}^{\infty} b_k(\beta)\mathrm{e}^{ikt}$$

where we have used Abel's theorem. We now have, by (1.5), (1.7) and Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} 2\pi\sum_{k=0}^{n} a_k\mathrm{e}^{ikx_0} = \int_{-\pi}^{\pi}\lim_{n\to\infty}\left(\sum_{k=0}^{n} b_k(\beta)\mathrm{e}^{ikt}\right)F(x_0-t)dt = \int_{-\pi}^{\pi}\frac{F(x_0-t)dt}{(1-\mathrm{e}^{it})^{1-\beta}}.$$

Hence $\sum_{k=0}^{\infty} a_k\mathrm{e}^{ikx_0}$ converges to $\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(x_0-t)dt}{(1-\mathrm{e}^{it})^{1-\beta}}$ and the theorem is proved.

Remark 1 Since $\mathcal{H}_\beta^2 = D_{2\beta}$, the special case $p=2$ of Theorem 1, with $\beta \in (0,1/2]$, deals with the convergence of $\sum_{k=0}^{\infty} a_k\mathrm{e}^{ikx}$ for functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in U for which $\sum k^{2\beta}|a_k|^2 < \infty$. It thus provides an alternative proof of the classical results of Beurling [4] and Salem and Zygmund ([15], [19, vol. 2, p. 195]) that in this case $\sum_{k=0}^{\infty} a_k\mathrm{e}^{ikx}$ converges outside exceptional sets of $C_{\beta,2}$-capacity zero. (If $\beta > 1/2$ and $\sum k^{2\beta}|a_k|^2 < \infty$ then $\sum|a_k| < \infty$ and $\sum_{k=0}^{\infty} a_k\mathrm{e}^{ikx}$ is convergent for all x.) Note that $C_{1/2,2}$-capacity corresponds to logarithmic capacity, and more generally that $C_{\beta,2}(E) = 0$ if and only if $C_{2\beta}(E) = 0$ where C_α denotes the classical α-capacity, $0 < \alpha \leq 1$ (see [2, Corollary 2.2]). Note also that the α-capacity used in [19] corresponds to $C_{1-\alpha}$. In section 2 we will apply Theorem 1 to derive *inter alia* some convergence results for $\sum_{k=0}^{\infty} a_k\mathrm{e}^{ikx}$ when $\sum k^\gamma|a_k|^p < \infty$ where $\gamma > 0$ and $p \in (1,\infty)$.

Remark 2 The classical results referred to above were proved for trigonometric series, but the power series formulation given the results here is easily shown to be equivalent. (See Section 2 below.)

Remark 3 Carleson [7, pp. 50-54] has proved the general result that if K is a convex kernel with $K(x) \equiv 0$ for $|x| \geq 1$, and $\sum_{k=1}^{\infty}\lambda_k|a_k|^2 < \infty$, where $(\lambda_k^{-1})_1^{\infty}$ is the (positive) sequence of Fourier cosine coefficients of K, then $\sum_{k=1}^{\infty} a_k\mathrm{e}^{ikx}$ converges outside a set E with $C_{\bar{K}}(E) = 0$, where $\bar{K}(r) = r^{-1}\int_0^r K(x)dx$ and $0 < r \leq 1$.

Remark 4 Since $\mathcal{H}_\beta^p$ is an analytic analogue in the unit disc for the class of Poisson integrals of Bessel potentials in half-spaces (see [13]), a natural analogue in $\mathbf{R}^n$ for the convergence question considered here for $\mathcal{H}^p_\beta$ is whether the spherical means

$$S_R f(x) = \int_{|\xi|\le R} \hat{f}(\xi) \mathrm{e}^{2\pi i x\cdot\xi} d\xi$$

converge as $R \to \infty$, where $\hat{f}$ denotes the Fourier transform of $f \in \mathcal{L}^p_\alpha(\mathbf{R}^n)$, the Sobolev space of convolutions of Bessel kernels with L^p functions in $\mathbf{R}^n, n \ge 2$. For known results and more detail see [6] and [12] and the cited references.

We will state and outline a proof of an extension of Theorem 1 in the final section below, and contrast our result with a theorem due to Temko [3, vol.1, pp.411-413], but we will focus in the next section on detailing some convergence consequences of Theorem 1 for two classes of analytic functions related to $\mathcal{H}^p_\beta$, and for certain trigonometric series.

2 *The classes $\mathcal{D}^p_\alpha$ and $\mathcal{B}^p_\gamma$. Trigonometric Series.*

For functions f analytic in U we write

$$\mathcal{D}^p_\alpha = \{f : \iint_U (1-|z|)^\alpha |f'(z)|^p \, dx\, dy < \infty\}, \quad \alpha > -1, p > 1,$$

and

$$\mathcal{B}^p_\gamma = \{f : f(z) = \sum_{k=0}^\infty a_k z^k, \ \sum_{k=0}^\infty k^\gamma |a_k|^p < \infty\}, \quad \gamma > 0, p > 1.$$

Note that for $p > 1, \mathcal{D}^p_\alpha$ is the weighted analytic Besov space (see [9], [5], [14], [18]), and that $D_{2\beta} = \mathcal{H}^2_\beta = \mathcal{B}^2_{2\beta} = \mathcal{D}^2_{1-2\beta}, 0 < \beta \le 1/2$. With a view to deducing some convergence results for the classes $\mathcal{D}^p_\alpha$ and $\mathcal{B}^p_\gamma$, we collect together in the following lemma some inclusion relations between these classes and $\mathcal{H}^p_\beta$.
Lemma *If* $1 < p \le 2$, *then*

$$\mathcal{D}^p_{p(1-\beta)-1} \subset \mathcal{H}^p_\beta, \quad \mathcal{B}^p_{p\beta} \subset \mathcal{H}^q_\beta, \tag{2.1}$$

where $0 < \beta < 1$ *and* $q = p/(p-1)$. *If* $p \ge 2$, *then*

$$\mathcal{D}^p_{1-p\beta} \subset \mathcal{B}^p_{p(1+\beta)-2} \subset \mathcal{H}^p_\beta \tag{2.2}$$

where $0 < \beta < 2/p$.

The first inclusion in (2.1) is established in [9] and the first inclusion of (2.2) follows from results in [14] (see also [5, Lemma 1.2]) for Bergman spaces. Note that $f \in \mathcal{D}^p_\alpha$ if and only if f' belongs to the Bergman space A^p_α. We derive the remaining results here.

We begin with the proof that $\mathcal{B}^p_{p\beta} \subset \mathcal{H}^q_\beta$ for $1 < p \leq 2, 0 < \beta < 1$. Assume that $f \in \mathcal{B}^p_{p\beta}$ and set $h_k = b_k(\beta)^{-1} a_k$ for $k \geq 0$, where $b_k(\beta)$ is defined as in the last section. Set $h_k = 0$ for $k < 0$. Then, using (1.3), $\sum_{-\infty}^{\infty} |h_k|^p \leq A_p \sum_{k=0}^{\infty} k^{p\beta}|a_k|^p < \infty$, so, by the Hausdorff-Young theorem [19, vol. 2, p. 101], (h_k) is the sequence of Fourier coefficients of a function $F \in L^q(-\pi, \pi), q = p/(p-1)$. Since $f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} b_k(\beta) h_k z^k$, it follows that $f \in \mathcal{H}^q_\beta$, which is the desired result.

We establish the inclusion $\mathcal{B}^p_{p(1+\beta)-2} \subset \mathcal{H}^p_\beta$ next. Suppose that $f \in \mathcal{B}^p_{p(1+\beta)-2}$, so that $\sum k^{p(1+\beta)-2}|a_k|^p$ is finite, with $0 < \beta < 1, p \geq 2$. Set $d_k = b_k(\beta)^{-1} a_k$ for $k \geq 0, d_k = 0$ for $k < 0$. Then

$$\sum_{-\infty}^{\infty} |k|^{p-2}|d_k|^p = \sum_{1}^{\infty} k^{p-2} b_k(\beta)^{-p}|a_k|^p \leq c_\beta \sum_{1}^{\infty} k^{p(1+\beta)-2}|a_k|^p < \infty,$$

and it follows by a result of Hardy and Littlewood [19, vol. 2, p. 110] that (d_k) is the sequence of Fourier coefficients of a function $F \in L^p(-\pi, \pi)$. Since $f(z) = \sum_{k=0}^{\infty} b_k(\beta) d_k z^k$ it follows that $f \in \mathcal{H}^p_\beta$, and we have obtained the second inclusion in (2.2) (for all $\beta \in (0, 1)$). This completes the proof of the lemma.

We now detail some of the convergence consequences that follow from combining Theorem 1 and the Lemma. From the first inclusion of (2.1) and Theorem 1, we deduce that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ is convergent $C_{\beta,p}$-quasieverywhere when $f \in \mathcal{D}^p_{p(1-\beta)-1}$ and $1 < p \leq 2$, $0 < \beta < 1$. In particular, choosing the values of β appropriately, we see that when $f \in \mathcal{D}^p_{p-2}$ and when $f \in \mathcal{D}^p_0$, $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ is respectively convergent $C_{1/p,p}$ and $C_{1/q,p}$-quasieverywhere on $[-\pi, \pi]$. (Note that if $\alpha p > \alpha' p'$, or if $\alpha p = \alpha' p'$ and $\alpha > \alpha'$, then [2] $C_{\alpha,p}(E) = 0$ implies $C_{\alpha',p'}(E) = 0$.) It follows from the second inclusion in (2.1), and Theorem 1, that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges $C_{\gamma/p,q}$-quasieverywhere when $f \in \mathcal{B}^p_\gamma$ and $1 < p \leq 2, 0 < \gamma \leq p-1$. Thus, for instance, if $\sum k^{p-1}|a_k|^p < \infty$ then $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges $C_{1/q,q}$-quasieverywhere. Note that if $f \in \mathcal{B}^p_\gamma$ and $p > 1$, then $\sum |a_k| < \infty$ when $q\gamma/p > 1$, *i.e.* when $\gamma > p - 1$.

From (2.2), setting $\beta = 1/p$ with $p \geq 2$, we obtain $\mathcal{D}^p_0 \subset \mathcal{B}^p_{p-1} \subset \mathcal{H}^p_{1/p}$ and hence $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges $C_{1/p,p}$-quasieverywhere for every $f \in \mathcal{D}^p_0 \cup \mathcal{B}^p_{p-1}$. The second inclusion in (2.2) gives $\mathcal{B}^p_\gamma \subset \mathcal{H}^p_{(\gamma+2-p)/p}$ for $p - 2 < \gamma \leq p - 1$ and $p \geq 2$, and the convergence results for $\mathcal{B}^p_\gamma$ which follow from this may be deduced from Theorem 1.

We can restate the results involving the class $\mathcal{B}^p_\gamma$ in terms of trigonometric series as follows:

If $p > 1, \sum k^\gamma(|\alpha_k|^p + |\beta_k|^p) < \infty$ and $A_k(x) = \alpha_k \cos kx + \beta_k \sin kx$, then the set of points of divergence of the trigonometric series $\sum A_k(x)$ is of $C_{\gamma/p,q}$-capacity zero, $q = p/(p-1)$, when $0 < \gamma \le p-1$ and $1 < p \le 2$, and of $C_{(\gamma+2-p)/p,p}$-capacity zero when $p-2 < \gamma \le p-1$ and $p > 2$.

(When $p = 2$ this coincides with the statement of the classical results of Salem and Zygmund in [19, vol. 2, p. 195].) To see that the results stated for trigonometric series are simply an equivalent reformulation of the results indicated above for the class $\mathcal{B}^p_\gamma$, note that if $p > 1$ and $a_k = \alpha_k - i\beta_k$ then

$$|\alpha_k|^p, |\beta_k|^p \le |a_k|^p = (|\alpha_k|^2 + |\beta_k|^2)^{p/2} \le (2\max(|\alpha_k|^2, |\beta_k|^2)^{p/2} \le 2^{p/2}(|\alpha_k|^p + |\beta_k|^p)$$

so $\sum n^\gamma(|\alpha_k|^p + |\beta_k|^p) < \infty \iff \sum n^\gamma |a_k|^p < \infty$. As $\sum \Re(a_k e^{ikx}) = \sum A_k(x)$ and $\sum \Im(a_k e^{ikx}) = \sum(-\beta_k \cos kx + \alpha_k \sin kx)$, the equivalence follows.

We finish this section by noting [17, Theorem 5] that if $p - 2 \le \alpha < p - 1$ and $p > 2$, then $\mathcal{D}^p_\alpha \subset \mathcal{B}^2_\gamma$ for every $\gamma < 2(p-\alpha-1)/p$ (but not, in general, for $\gamma = 2(p-\alpha-1)/p$), with consequent convergence implications for $\sum a_k e^{ikx}$ by the classical case $p = 2$ of Theorem 1 above. In particular, if $f \in \mathcal{D}^p_{p-2}$ and $p > 2$, then $\sum k^\gamma |a_k|^2 < \infty$, and $\sum a_k e^{ikx}$ is convergent $C_{\gamma/2,2}$-quasieverywhere, for every $\gamma < 2/p$.

3 *An extension of Theorem 1*

We begin by recalling some standard notation relating to sequences. For a real sequence $(d_k)_0^\infty$ we write $\Delta d_k = d_k - d_{k+1}$, so that (d_k) is decreasing if $\Delta d_k \ge 0$. We say that (d_k) is convex if $\Delta^2 d_k = \Delta(\Delta d_k) \ge 0$. We suppose now that $(\lambda_k)_0^\infty$ is a positive convex sequence such that $\lambda_k \to 0$ and $\sum \lambda_k = \infty$, and is also such that the sequence $(\gamma_k)_1^\infty$ decreases to zero as $k \to \infty$ where $\gamma_k = k\Delta\lambda_k$ for $k \ge 1$. We write

$$\Gamma_n(x) = \sum_{k=1}^n \gamma_k e^{ikx},\ n \ge 1,$$

and define

$$K_\lambda(x) = \int_1^{\pi/|x|} t[\lambda(t) - \lambda(t+1)]dt, \quad 0 < |x| \le \pi, \tag{3.1}$$

where $\lambda(t)$ is interpolated linearly between $\lambda_k = \lambda(k)$ and $\lambda_{k+1} = \lambda(k+1)$. We then have from [3, vol. 1, pp. 408-9] that

$$|\Gamma_n(x)| \leq B_1 K_\lambda(x) + B_2,\ 0 < |x| \leq \pi, \tag{3.2}$$

where B_1, B_2 are positive quantities independent of n. The inequality (3.2) is established in [3] with $\Gamma_n(x) = \sum_{k=1}^n \gamma_k \cos kx$ but the proof is easily seen to apply to the sum $\sum_{k=1}^n \gamma_k \sin kx$ as well, so the result stated here follows. We extend the definition of K_λ to $\mathbf{R}$ by setting $K_\lambda(x) \equiv 0$ for $|x| > \pi$. Then K_λ is a kernel: it is even, decreasing, unbounded since

$$K_\lambda(x) \geq \sum_{k=1}^{[\pi/|x|]} k[\lambda_{k+1} - \lambda_{k+2}] \to \infty, \quad x \to 0,$$

because $\sum \lambda_k = \infty$ (see [3, vol. 1, p. 6]), and is integrable on $\mathbf{R}$ since

$$\int_0^\pi K_\lambda(x)dx = \pi \int_1^\infty [\lambda(t) - \lambda(t+1)]dt \leq \pi \sum_{k=1}^\infty [\lambda_k - \lambda_{k+1}] = \pi\lambda_1.$$

Consequently, by a simple property of kernels noted in the first section above, we have, for every F in $L^p(\mathbf{R})$, that

$$\int_{\mathbf{R}} K_\lambda(t)\,|F(x-t)|\,dt < \infty \tag{3.3}$$

$C_{K_\lambda,p}$-quasieverywhere in $\mathbf{R}$.

We are now in a position to state our final theorem.

Theorem 2 *Let the sequences (λ_k) and (γ_k) and the kernel K_λ be defined as detailed above. Define $g(z) = \sum_{k=0}^\infty \gamma_k z^k$ for $z \in U$ and set*

$$f(z) = \sum_{k=0}^\infty a_k z^k = \int_{-\pi}^{\pi} g(z\mathrm{e}^{-it})F(t)dt, \ \ z = r\mathrm{e}^{ix} \in U, \tag{3.4}$$

where $F \in L^p(-\pi, \pi)$ is extended by periodicity to $[-3\pi, 3\pi]$ and $F \equiv 0$ in $\mathbf{R}\backslash[-3\pi, 3\pi]$. Then $\sum_{k=0}^\infty a_k \mathrm{e}^{ikx}$ converges $C_{K_\lambda,p}$-quasieverywhere on $[-\pi, \pi]$ to $\int_{-\pi}^{\pi} g(\mathrm{e}^{-i(t-x)})F(t)dt$.

To prove Theorem 2 it is enough to show that $\sum_{k=0}^\infty a_k \mathrm{e}^{ikx}$ converges for every x for which $K_\lambda * |F|(x)$ is finite. To do this we apply an argument similar to the one used above to obtain Theorem 1, replacing (1.6) with (3.2), and using (3.3) instead of (1.4). We omit the details which are easily provided.

Theorem 2 is a generalization of Theorem 1. To see this, choose $\gamma_k = b_k(\beta)$ and $\lambda_k = \sum_{m=k}^{\infty} b_m(\beta)/m, k \geq 1$, in Theorem 2. Then $g(z) = (1-z)^{\beta-1}$ and, writing N_x for $[\pi/|x|]$, we have from (3.1) that

$$\sum_{k=1}^{N_x} k\Delta\lambda_{k+1} \leq K_\lambda(x) \leq \sum_{k=1}^{N_x+1} (k+1)\Delta\lambda_k, \tag{3.5}$$

and so

$$\frac{1}{2}\sum_{k=1}^{N_x} b_{k+1}(\beta) \leq K_\lambda(x) \leq 2\sum_{k=1}^{N_x+1} b_k(\beta).$$

Using (1.3) and (1.1), we easily obtain $K_\lambda(x) \sim G_\beta(x), x \to 0$. Hence Theorem 2 contains Theorem 1.

We illustrate Theorem 2 with some examples. For our first example we assume that $\sum k^{p-2}(\log(k+1))^{p+\varepsilon}|a_k|^p < \infty$ where $p \geq 2$ and $\varepsilon > 0$. We set

$$\gamma_k = k\Delta\lambda_k = (\log(k+1))^{-1-\varepsilon/p},\ k \geq 1,$$

so that

$$\lambda_k = \sum_{m=k}^{\infty} \frac{\gamma_m}{m} = \sum_{m=k}^{\infty} 1/m(\log(m+1))^{1+\varepsilon/p} \sim (\log(k+1))^{-\varepsilon/p},$$

giving $\sum \lambda_k = \infty$. Then, using (3.5), we deduce that

$$K_\lambda(x) \sim \sum_{k=1}^{N_x}(\log(k+1))^{-1-\varepsilon/p} \sim 1/|x|(\log(\pi/|x|))^{1+\varepsilon/p} \tag{3.6}$$

as $x \to 0$. We note next that

$$\sum k^{p-2}\gamma_k^{-p}|a_k|^p = \sum k^{p-2}\log(k+1)^{p+\varepsilon}|a_k|^p < \infty,$$

by assumption, and it follows (see proof of second inclusion (2.2)) that $(d_k)_{-\infty}^{\infty}$, with $d_k = \gamma_k^{-1}a_k$ for $k \geq 1$ and 0 otherwise, is the sequence of Fourier coefficients of a function $F \in L^p(-\pi,\pi)$. Since $a_k = \gamma_k d_k,\ k \geq 1$, we have that $f(z) = \sum a_k z^k$ is of the form (3.4), with $g(z) = \sum \gamma_k z^k$, and Theorem 2 now yields the result that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges $C_{K_\lambda,p}$-quasieverywhere where K_λ satisfies (3.6).

Next, as a second example, we assume that $\sum k^{2-p}(\log(k+1))^{p+\varepsilon}|a_k|^p < \infty$, with $1 < p < 2$ and $\varepsilon > 0$, and set $\gamma_k = k^{1-2/p}(\log(k+1))^{-1-\varepsilon/p}$. Then

$$\lambda_k = \sum_{m=k}^{\infty} 1/m^{2/p}(\log(m+1))^{1+\varepsilon/p} \sim k^{1-2/p}/(\log(k+1))^{1+\varepsilon/p},$$

as $k \to \infty$, and $\sum \lambda_k = \infty$ since $p > 1$. Next, writing $d_k = \gamma_k^{-1} a_k$ again, we have

$$\sum |d_k|^p = \sum \gamma_k^{-p} |a_k|^p = \sum k^{2-p} (\log(k+1))^{p+\varepsilon} |a_k|^p < \infty.$$

Hence, using the Hausdorff-Young theorem, we find that $f(z) = \sum a_k z^k$ is of the form (3.4) for a function $F \in L^q(-\pi, \pi), q = p/(p-1)$. Consequently, $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges outside a set of $C_{K_\lambda,q}$ capacity zero where, by (3.5),

$$K_\lambda(x) \sim 1/|x|^{2/q} (\log(\pi/|x|))^{1+\varepsilon/p}.$$

The theorem of Temko [3, vol. 1, p. 411] referred to above, which deals only with the case $p = 2$, and involves exceptional sets of an appropriate *generalised convex capacity* zero, yields the result that $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges outside an exceptional set of capacity zero if $\sum (\log(k+1))^{1+\varepsilon} |a_k|^2 < \infty$ for any $\varepsilon > 0$ (but not for $\varepsilon = 0$), a stronger result than the corresponding result obtained in the examples above from Theorem 2 when $p = 2$. Indeed, Temko has shown more generally that if $\sum W(k)|a_k|^2 < \infty$, where $(W(k))$ is an increasing sequence such that $\sum 1/kW(k) < \infty$ then $\sum_{k=0}^{\infty} a_k \mathrm{e}^{ikx}$ converges outside a set of convex capacity zero. For more detail, see [3].

We conclude by noting that it was proved in the Kolmogorov-Seliverstov-Plessner theorem [3, vol. 1, p. 364] that if $\sum \log(k+1)|a_k|^2 < \infty$ then the series $\sum a_k \mathrm{e}^{ikx}$ is convergent *almost everywhere* in $[-\pi, \pi]$, a result that was strengthened by L. Carleson in [8] who proved the long-standing conjecture of Lusin that $\sum |a_k|^2 < \infty$ is sufficient to ensure almost everywhere convergence of $\sum a_k \mathrm{e}^{ikx}$.

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